Konuralp Journal of Mathematics, 13 (2) (2025) 300-304

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

On the Existence of Fixed Points for Weak Hicks Contractions in Generalized b-Menger Spaces

Abderrahim Mbarki[1*] and Mohammed Elberkani[2]

[1]*National school of Applied Sciences, P.O. Box 669, Mohammed First University Oujda, Morocco.*
[2]*ANO Laboratory, Faculty of Sciences, Mohammed First University, Oujda, Morocco.*
[*]*Corresponding author*

Abstract

This paper investigates the existence of fixed points for weak Hicks contractions in generalized b-Menger spaces. An example is provided to illustrate the validity of the results obtained.

Keywords: *Fixed Point; Generalized b-Menger Space; Weak Hick Contraction.*
2010 Mathematics Subject Classification: *Primary 54E70; 54H25; Secondary 47S50; 34B15.*

1. Introduction and preliminaries

In 2017, Mbarki et al. [7] introduced the concept of the probabilistic b-metric, a novel extension of the standard probabilistic metric, which generalizes the probabilistic triangle inequality. This pioneering work established the foundation for numerous advancements in the field, leading to several significant fixed-point results across various settings, as highlighted in their subsequent publications [1], [6], [8, 13].
More recently, Oubrahim et al. [15] introduced the concept of generalized b-Menger spaces, which extends the definition of b-Menger spaces by replacing the constant s with a controlled function in the triangle inequality.
Building on these developments, the present paper aims to advance the theory of fixed-point theorems within the framework of generalized b-Menger spaces. Specifically, we investigate the existence of fixed points for weak-Hicks contraction maps. To illustrate the key concepts, we provide several non-trivial examples, which not only demonstrate the theoretical results but also highlight their practical relevance.
The following sections offer a brief overview of the essential definitions and notations that will be employed throughout the paper.

Definition 1.1. *A distance distribution function (abbreviated as d.d.f.) is a non-decreasing function F defined on $\mathbb{R}^+ \cup \{+\infty\}$) that satisfies the conditions $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0, +\infty)$. The collection of all such functions will be denoted by Δ^+, while the subset of Δ^+ for which $\lim_{t\to+\infty} F(t) = 1$ will be denoted by D^+ .*

As a simple illustration, the Heaviside function is an example of a distribution function in D^+, which is defined as follows:

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

A *t-norm* is a commutative, associative, and non-decreasing operation $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following properties:

1. $T(a,1) = a$ for all $a \in [0,1]$,
2. $T(0,0) = 0$.

As common examples of continuous t-norms, we mention the following:

- $T_p(a,b) = ab$,
- $T_M(a,b) = \min(a,b)$,
- $T_L(a,b) = \max\{a+b-1, 0\}$.

Email addresses: dr.mbarki@gmail.com (Abderrahim Mbarki), mohammed.elberkani@gmail.com (Mohammed Elberkani)

For brevity, we will write $T^1(x,y) = T(x,y)$, $T^2(x,y,z) = T(x,T^1(y,z))$, and for $n \geq 3$, we define

$$T^n(x_1,x_2,\ldots,x_{n+1}) = T(x_1,T(x_2,x_3,\ldots,x_{n+1})).$$

A t-norm T can also be extended to an infinite countable operation. For any sequence $\{x_n\}$ in $[0,1]$, we define:

$$T_{i=1}^{\infty} x_i = \lim_{n\to\infty} T^n(x_1,\ldots,x_{n+1}),$$

which always exists because the sequence $\{T^n(x_1,\ldots,x_{n+1})\}$ is non-decreasing and bounded from below. Additionally, we have:

$$T_{i=n}^{\infty} x_i = T_{i=1}^{\infty} x_{n+i}.$$

We will frequently make use of the following definitions throughout this paper.

Definition 1.2. *([2]) We say that a t-norm T is of H-type if the family $\{T^n(t)\}$ is equicontinuous at $t = 1$, that is,*

$$\forall \varepsilon \in (0,1) \quad \exists \lambda \in (0,1) : t > 1-\lambda \Rightarrow T^n(t) > 1-\varepsilon \quad \text{for all} \quad n \geq 1,$$

Definition 1.3. *([3]) A t-norm T is geometrically convergent (g-convergent) if, for all $q \in (0,1)$,*

$$\lim_{n\to\infty} T_{i=n}^{\infty}(1-q^i) = 1$$

It should be noticed (see [3]) that if, $\lim_{n\to\infty} T_{i=n}^{\infty}(1-q_0{}^i) = 1$ for some $q_0 \in (0,1)$, then T is g-convergent t-norm. Also, note that if T is g-convergent then $\sup_{a<1} T(a,a) = 1$. Some sufficient condition for T we given by Hadžić et al. [3] to ensure that T is a g-convergent t-norm. Also, note that T_L and the t-norms of H-type are g-convergent t-norms.

Definition 1.4. *([15]) A generalized b-Menger space is a quadruple (M,F,T,α) where M is a nonempty set, F is a function from $M \times M$ into Δ^+, T is a t-norm, $\alpha : M \times M \to [1,\infty)$, and the following conditions are satisfied:*
For all $p,r,q \in M$ and $x,y > 0$,

1. $F_{pp} = H$,
2. $F_{pr} = H \;\Rightarrow\; p = r$,
3. $F_{pr} = F_{rp}$,
4. $F_{pr}(\alpha(p,r)(x+y)) \geq T(F_{pq}(x),F_{qr}(y))$.

It is important to note that a b-Menger space is a special case of a generalized b-Menger space, where $\alpha(p,r) = 1$ for all $p,r \in M$.

Definition 1.5. *Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (M,F) (i.e., conditions (1), (2), and (3) of Definition 1.4 are satisfied)*

1. *A sequence $\{x_n\}$ in M is said to converge to x in M if, for every $\varepsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $N(\varepsilon,\delta)$ such that $F_{x_n x}(\varepsilon) > 1-\delta$ whenever $n \geq N(\varepsilon,\delta)$.*
2. *A sequence $\{x_n\}$ in M is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $N(\varepsilon,\delta)$ such that $F_{x_n x_m}(\varepsilon) > 1-\delta$ whenever $n,m \geq N(\varepsilon,\delta)$.*
3. *(M,F) is said to be complete if every Cauchy sequence has a limit in M.*

2. Main result

The class of probabilistic contractions, which has been extensively studied in the context of fixed point theory for probabilistic metric spaces (see, e.g., [4], [14]), was introduced by T. L. Hicks in [5]. The definition is as follows

Definition 2.1. *A mapping f of a probabilistic semimetric space (M,F) into itself is called a Hicks contraction (or C-contraction for short) if there exists a constant $k \in (0,1)$ such that for all $p,q \in M$ and for every $t > 0$, the following implication holds:*

$$(H): \quad F_{pq}(t) > 1-t \quad \Longrightarrow \quad F_{fpfq}(kt) > 1-kt.$$

Miheţ ([14]) studied a generalization of the C-contraction, referred to as the weak-Hicks contraction

Definition 2.2. *A mapping f of a probabilistic semimetric space (M,F) into itself is called a weak-Hicks contraction (or $w-H$ contraction for short) if there exists a constant $k \in (0,1)$ such that for all $p,q \in S$, the following condition holds:*

$$(w-H): \quad \text{if} \quad t \in (0,1),\ F_{pq}(t) > 1-t, \quad \text{then} \quad F_{f(p)f(q)}(kt) > 1-kt.$$

We can now present and prove the main fixed point theorem of this paper.

Theorem 2.3. *Let (M,F,T,α) be a complete generalized b-Menger space such that T is a g-convergent t-norm, and let f be a weak-Hicks contraction with $k \in (0,1)$. Suppose further that for all $x \in M$, we have*

$$\sup_{n\in\mathbb{N}^*,\ j\in\mathbb{N}} \alpha(x_n,x_{n+j}) < \frac{1}{k},$$

where $x_n = f^n x$ and $x_0 = f^0 x = x$. Then f has a fixed point if and only if there exists $x \in M$ such that $F_{xfx}(1) > 0$.

Proof. Assume that f has a fixed point x. Then, we have $F_{xfx}(1) = F_{xx}(1) = 1 > 0$.
For the converse implication, we first show that there exists a $\delta \in (0,1)$ such that $F_{xfx}(\delta) > 1-\delta$. Indeed, suppose that $F_{xfx}(\delta) \leq 1-\delta$ for all $\delta \in (0,1)$. Taking the limit as $\delta \to 1$ and using the fact that F_{xfx} is left-continuous, we obtain $F_{xfx}(1) \leq 0$, which leads to a contradiction. Therefore, there must exist a $\delta \in (0,1)$ such that $F_{xfx}(\delta) > 1-\delta$.
Now, applying the weak-Hicks contraction property (w-H), we can prove by induction that

$$F_{f^n x f^{n+1} x}(k^n \delta) > 1 - k^n \delta$$

for every $n \in \mathbb{N}$.

We will prove that the sequence $\{x_n\}$ is a Cauchy sequence, i.e., that for every $\varepsilon, \gamma > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$F_{x_n x_{n+m}}(\varepsilon) > 1 - \gamma$$

for all $n \geq n_0$ and for every $m \in \mathbb{N}$.
Since T is a g-convergent t-norm, then exists $n_1 \in \mathbb{N}$ such that

$$T_{i=n}^{\infty}(1-k^i) > 1-\gamma$$

for all $n \geq n_1$. Let $n_0 \in \mathbb{N}$ be such that

$$\sum_{i=0}^{m-1} \prod_{j=0}^{i} [\alpha(x_{n+j}, x_{n+m})(k)^{n+j}]\delta \leq k^n \delta \sum_{i=0}^{\infty} \rho^i < \varepsilon,$$

for all $n \geq n_0$, with $\rho = k \sup_{n \in \mathbb{N}^*,\ j \in \mathbb{N},} \alpha(x_n, x_{n+j}) < 1$. Then for $n \geq n"_0 = \max\{n_0, n_1\}$, we have

$$\begin{aligned}
F_{x_n x_{n+m}}(\varepsilon) \geq & F_{x_n x_{n+m}}(\sum_{i=0}^{m-1} \prod_{j=0}^{i} [\alpha(x_{n+j}, x_{n+m})(k)^{n+j}]\delta) \\
& \geq T(F_{x_n x_{n+1}}(k^n\delta), F_{x_{n+1} x_{n+m}}(\sum_{i=1}^{m-1} \prod_{j=1}^{i} [\alpha(x_{n+j}, x_{n+m})(k)^{n+j}]\delta) \\
& \vdots \\
& \geq T^{m-1}(F_{x_n x_{n+1}}(k^n\delta), F_{x_{n+1} x_{n+2}}(k^{n+1}\delta), ..., F_{x_{n+m-1} x_{n+m}}(k^{n+m-1}\delta)) \\
& \geq T^{m-1}(1-k^n\delta, ..., 1-k^{n+m-1}\delta) \\
& \geq T^{m-1}(1-k^n, ..., 1-k^{n+m-1}) \\
& \geq T_{i=n}^{\infty}(1-k^i) \\
& > 1-\gamma.
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in (M, F, T, α). By completeness, there exists some $z \in M$ such that $x_n \to z$. Now, we will show that $x_n \to fz$. Indeed, given $\varepsilon > 0$ and $\lambda \in (0,1)$, we choose $\sigma \in (0,1)$ such that $0 < k\sigma < \min\{\varepsilon, \lambda\}$. Then, there exists an integer N such that for all $n \geq N$, we have

$$F_{x_n z}(\sigma) > 1-\sigma \implies F_{f x_n f z}(k\sigma) > 1-k\sigma \implies F_{f x_n f z}(k\sigma) > 1-\lambda \implies F_{f x_n f z}(\varepsilon) > 1-\lambda.$$

Thus

$$\lim_{n \to \infty} x_{n+1} = fz.$$

By inequality (4) in Definition 1. 4, we get

$$F_{fzz}(t) \geq T(F_{f z x_{n+1}}(\frac{t}{2\alpha(fz,z)}), F_{x_{n+1} z}(\frac{t}{2\alpha(fz,z)})),$$

for all $t > 0,\ n \geq 1$.
Since $\sup_{a<1} T(a,a) = 1$, let $\sigma \in (0,1)$, and there exists $\rho \in (0,1)$ such that

$$T(1-\rho, 1-\rho) > 1-\sigma.$$

Using the fact that $F_{f z x_{n+1}}\left(\frac{t}{2\alpha(fz,z)}\right) \to 1$ and $F_{x_{n+1} z}\left(\frac{t}{2\alpha(fz,z)}\right) \to 1$, there exists an integer $N \in \mathbb{N}$ such that

$$F_{f z x_{n+1}}\left(\frac{t}{2\alpha(fz,z)}\right) > 1-\rho, \quad F_{x_{n+1} z}\left(\frac{t}{2\alpha(fz,z)}\right) > 1-\rho,$$

for all $n \geq N$. Therefore, we have

$$\begin{aligned}
F_{fzz}(t) &\geq T(F_{f z x_{n+1}}(\frac{t}{2\alpha(fz,z)}), F_{x_{n+1} z}(\frac{t}{2\alpha(fz,z)})) \\
&\geq T(1-\rho, 1-\rho) \\
&\geq 1-\sigma.
\end{aligned}$$

It follows that

$$F_{fzz}(t) \geq 1-\sigma, \forall \sigma \in (0,1).$$

Letting $\sigma \to 0$, we obtain $F_{fzz}(t) = 1$. Since this holds for any arbitrary $t > 0$, it follows that $fz = z$. □

It should be noted, unlike in previous works [7, 15], that in the proof of Theorem 2.3, the conditions $Range(F) \subset D^+$, and that T is an H-type or continuous t-norm on the generalized b-Menger metric, are not required.
By taking $\alpha(x,y) = s$ for all $x,y \in M$ in Theorem 2.3, we obtain the following result

Corollary 2.4. *([6]) Let (M,F,T) be a complete b-Menger space such that T is a g-convergent t-norm and f be a w-H contraction. Then f has a fixed point iff there exists $x \in M$ such that $F_{xfx}(1) > 0$.*

We illustrate the results from Theorem 2.3 with the following example:

Example 2.5. *Let $M = [0,\infty)$. Define $F : M \times M \to \Delta^+$ by*

$$F_{pq}(t) = \begin{cases} H(t) & \text{if} \quad p = q, \\ H(t - e^{|p-q|}) & \text{otherwise.} \end{cases}$$

Consider the mapping $\alpha : M \times M \to [1,+\infty)$ defined as follows:

$$\alpha(p,q) = e^{|p-q|}.$$

We claim that $(M,\ F,\ T_L,\ \alpha)$ is a generalized b-Menger space. Indeed, if

$$T_L(F_{pr}(t), F_{rq}(s)) = \max\{F_{pr}(t) + F_{rq}(s) - 1, 0\} > 0$$

this implies that $H(t - e^{|p-r|}) = H(s - e^{|r-q|}) = 1$, then $t > e^{|p-r|}$ and $s > e^{|r-q|}$. So,

$$\begin{aligned} e^{|p-q|}(s+t) &> e^{|p-q|}(e^{|p-r|} + e^{|r-q|}) \\ &\geq e^{|p-q|}. \end{aligned}$$

Hence

$$\begin{aligned} F_{pq}(\alpha(p,q)(s+t)) &\geq H((s+t)e^{|p-q|} - e^{|p-q|}) \\ &\geq 1 \\ &\geq T_L(F_{pr}(t), F_{rq}(s)) \end{aligned}$$

which prove our claim. It is obvious that $(M,\ F,\ T_L,\ \alpha)$ is complete because any Cauchy sequence in M must eventually be constant. Now consider the function $f :\ M \to M$ given by

$$f(p) = \begin{cases} 0 & \text{if} \quad p = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since $\sup_{n\in\mathbb{N}^*, j\in\mathbb{N}} \alpha(x_n, x_{n+j}) = 1$ for all $x \in M$, $F_{0f0}(1) > 0$, and for any $t \in (0,1)$, if $F_{pq}(t) > 1 - t$ implies $p = q$, it follows that $F_{f(p)f(q)}(kt) = 1$. Thus, f is a weak Hicks contraction. Clearly, all the hypotheses of Theorem 2.3 are satisfied, and the mapping has two fixed points: 0 and 1. However, we cannot apply the results from [6] here, because (M,F,T_L) is not a standard b-Menger space due to the absence of the triangle inequality. To illustrate this, suppose there exists $s \geq 1$ such that

$$F_{pq}(s(x+y)) \geq T_L(F_{pr}(x), F_{rq}(y)) \text{ for all } x,y > 0,\ p,\ r,q \in M.$$

So,

$$x > e^{|p-r|} \text{ and } y > e^{|r-q|},$$

implies that

$$s(x+y) > e^{|p-q|}.$$

Hence

$$e^{|p-q|} < s(e^{|p-r|} + e^{|r-q|} + \varepsilon),$$

for all $\varepsilon > 0,\ p,\ r,q \in M$. Letting $\varepsilon \to 0$, we get

$$e^{|p-q|} \leq s(e^{|p-r|} + e^{|r-q|}),$$

for all $p,\ r,q \in M$. Now let $t > 0$ and take $p = 5t, q = t, r = 3t$ from above we get

$$e^{4t} \leq s(e^{2t} + e^{2t}),$$

then

$$e^{2t} \leq 2s,$$

for all $t > 0$, which leads to a contradiction. On the other hand, we cannot apply the results from [15], since T_L is not a t-norm of H-type.

3. Conclusion

This paper establishes the existence of fixed points for weak Hicks contractions in generalized b-Menger spaces, offering new results. Unlike previous studies, the proof of Theorem 2.3 does not require conditions such as T being an H-type t-norm or continuous. However, the results cannot be directly applied to earlier works like [15], as T_L is not an H-type t-norm. The example provided demonstrates the validity of the findings and suggests potential directions for future research.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Copyright Statement: Authors own the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

References

[1] Y. Achtoun, S. Radenović, I. Tahiri, M. L. Sefian, The nonlinear contraction in probabilistic cone b-metric spaces with application to integral equation , Nonlinear Analysis: Modelling and Control, Vol. 29, No. 3, (2024) 1-12.
[2] O. Hadžić, A fixed point theorem in Menger spaces, Publ. Inst. Math. (Beograd) T. 20 ,(1979), 107-112.
[3] O. Hadžić, E. Pap, M. Budinccvic, Countable extension of triangular norms and their applications to fixed point theory in probabilistic metric spaces, Kibermetica, 38, 3 (2002) 363-381.
[4] O. Hadžić, E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluwer Academic Publishers, Dordrecht (2001).
[5] T.L. Hicks, Fixed point theory in PM spaces, Rev. Resh. Novi. Sad., 13, (1983) 63-72.
[6] A. Mbarki, J. Hlal, Weakly-H contraction fixed point theorem in b-menger spaces, International Journal of Applied Mathematics, 35, No 2 (2022), 225–232.
[7] A. Mbarki, R. Oubrahim, Probabilistic b-metric spaces and nonlinear contractions, Fixed Point Theory and Applications, 2017, vol. 2017, no 1, p. 29.
[8] A. Mbarki, R. Oubrahim, Fixed point theorems with cyclical contractive conditions in b-Menger spaces, Results in Fixed Point Theory and Applications, 2018, (2018); doi: 10.30697/rfpta-2018-005.
[9] A. Mbarki, R. Oubrahim, *Common fixed point theorems in b-Menger spaces*, In: Recent Advances in Intuitionistic Fuzzy Logic Systems. Studies in Fuzziness and Soft Computing, Springer, (2019), 283-289.
[10] A. Mbarki, R. Oubrahim, Common fixed point theorem in *b*-Menger spaces with a fully convex structure, Int. J. Appl. Math., 32, No 2, (2019), 219-238; doi: 10.12732/ijam.v32i5.1.
[11] A. Mbarki, R. Oubrahim, Fixed point theorem satisfying cyclical conditions in b-Menger spaces, Moroccan J. Pure. Appl. Anal., 5, No 1, (2019), 31-36.
[12] A. Mbarki, R. Oubrahim, Probabilistic φ-contraction in *b*-Menger spaces with fully convex structure, Int. J. Appl. Math., 33, No 4 (2020), 621-633.
[13] A. Mbarki, R. Oubrahim, *Some properties of convexity structure and applications in b-Menger spaces*, In : Mathematical and Computational Methods for Modelling, Approximation and Simulation. Cham : Springer International Publishing, 2022. p. 181-189.
[14] D. Miheţ, Weak-Hicks contractions, Fixed Point Theory, 6, No 1, (2005), 71-78.
[15] R. Oubrahim, R. Naciri, A. Mbarki, Fixed point theorems in generalized b-Menger spaces, Results in Nonlinear Analysis, 7 (2024) No. 1, 35-43.
[16] B. Schweizer, H. Sherwood and R.M. Tradif, Contractions on probabilistic metric spaces: examples and counterexamples, Stochastica., XII, 1 (1988), 5-17.